Relational verification using product programs

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First PROMETIDOS Summer School, UCM, Madrid

Relational Program Reasoning

Captures the fact that two programs behave similarly or that the same program behaves similarly in two different executions.

- Compiler correctness
- Program continuity
- Non-interference

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Translation validation

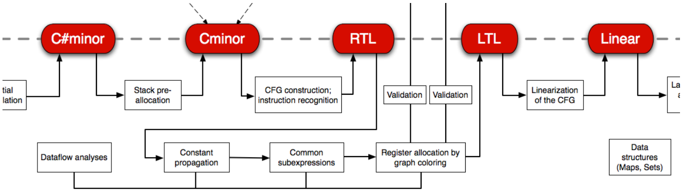

image fragment taken from the Compcert webpage.

Several techniques to provide trust over compiled code:

- Translation validation
- Certified compilation
- Certificate translation

Definition

Validating a program transformation consists on establishing that, assuming correct the result of the analysis, the original program and the transformed program are observationally equivalent.

Translation validation

For each compilation step: $c_i \longrightarrow c_{i+1}$

$$\begin{array}{llccc} \textbf{Execution} & c_i : & s_1 & \rightsquigarrow & s_1' \\ \textbf{Execution} & c_{i+1} : & s_2 & \rightsquigarrow & s_2' \end{array}$$

if the initial states s_1 and s_2 are observationally equivalent, then the final states s_1' and s_2' are observationally equivalent:

$$\forall x. s_1 x = s_2 x \Longrightarrow \forall x. s_1' x = s_2' x$$

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Translation validation

Example: are these programs equivalent?

```
i := 0;                i := 0;
                       j := C;
while (i < N) do       while (i < N) do
   j := i*B+C;            x += j;
   x += j;                j += B;
   i++                    i++
```

lots of work on translation validation

- Pnueli, Zuck et al. Translation validation
- Sorin Lerner et al. Rhodium tool
- Benton's RHL

Translation validation

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$i := 0$;

while ($i < N$) do
    $j := i * B + C$;
    $x$ += $j$;
    $i$++
```

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$i := 0$;
$j := C$;
while ($i < N$) do
    $x$ += $j$;
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    $i$++
```

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Program continuity

Analysis of program robustness under small input variations

[Gulwani et al.]

Given a program c:

$$\textbf{Execution } c : \quad s_1 \quad \rightsquigarrow \quad s_1'$$

$$\textbf{Execution } c : \quad s_2 \quad \rightsquigarrow \quad s_2'$$

if s_1 and s_2 differ in infinitesimal values, then s_1' and s_2' differ in infinitesimal values. For any ϵ, there is δ s.t.:

$$\forall x. \ \mid s_1 x - s_2 x \mid < \delta \Longrightarrow \forall x. \ \mid s_1' x - s_2' x \mid < \epsilon$$

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Program continuity

Example: is bubblesort continuous?

```
$i := 0$;
$j := N$;
while $(i \leq N)$ do
    $j := N$;
    while $(j > i)$ do
        if $(a[j-1] > a[j])$ then
            $x := a[j]$;
            $a[j] := a[j-1]$;
            $a[j-1] := x$
        $j := j-1$;
    $i := 1+i$
```

Non-interference

Basic idea

Equal low level inputs produce the same low outputs, regardless of the high level inputs.

Given a program c:

$$\textbf{Execution } c: \quad s_1 \quad \rightsquigarrow \quad s_1'$$

$$\textbf{Execution } c: \quad s_2 \quad \rightsquigarrow \quad s_2'$$

if s1 and s2 coincide in the public portion of the state, then s1' and s2' coincide in the public portion of the state.

$$\forall x \in L.\ s_1 x = s_2 x \Longrightarrow \forall x \in L.\ s_1' x = s_2' x$$

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Basic idea

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If it looks like we are talking about the same thing is in fact because we are.

Hyperproperties

All these relational properties are particular instances of Clarkson and Schneider *Hyperproperties*: Journal of Computer Security (2010)

Property: a set of execution traces

Hyperproperty: a set of sets of execution traces
(or a set of trace properties)
(or a property of sets of traces)

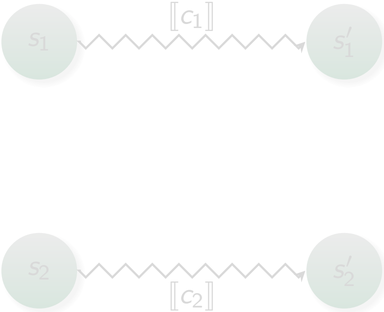
In particular k-safety properties: the bad thing never involves more than k traces.

program equivalence, program continuity and non-interference are 2-safety properties.

RELATIONAL JUDGEMENTS

Relational judgement

$$\vDash \{\varphi\} c_1 \sim c_2 \{\psi\}$$



Relational judgement

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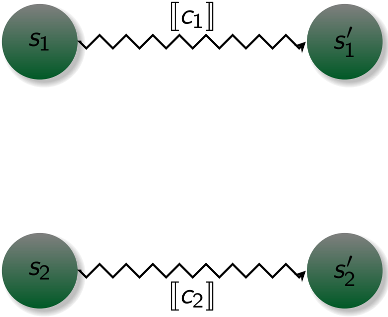

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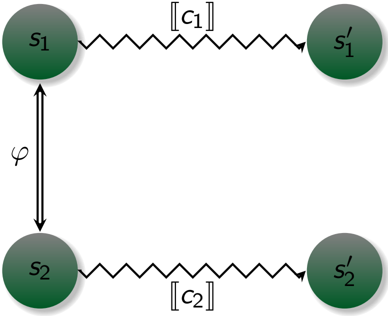

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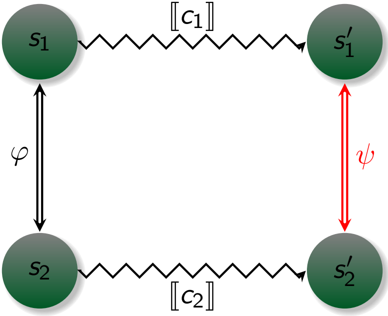

Give me a method to verify these relational judgements, and I'll have the power to prove the 2-safety properties we've been mentioning before.

Program transformations

$$\varphi \doteq \bigwedge_{x_i \in \mathbf{FV}(c_1)} x_i = x_i' \qquad \psi \doteq \bigwedge_{x_i \in \mathbf{FV}(c_1)} x_i = x_i'$$

Program continuity

$$\varphi \doteq \bigwedge_{x_i \in \mathbf{FV}(c_1)} |x_i - x_i'| < \delta \qquad \psi \doteq \bigwedge_{x_i \in \mathbf{FV}(c_1)} |x_i - x_i'| < \epsilon$$

Non-interference

$$\varphi \doteq \bigwedge_{x_i \in L} x_i = x_i' \qquad \psi \doteq \bigwedge_{x_i \in L} x_i = x_i'$$

A variety of logical methods have been proposed independently for each particular problem, as they were no connection between them.

Self-composition.

Logical verification of non-interference.

- typing and static analyses reject many secure programs
- logical frameworks extend the expressivity of declassification

Partial release (Sabelfeld and Myers)

$$\{(h \geq k)_{\langle 1\rangle} = (h \geq k)_{\langle 2\rangle} \wedge l_{\langle 1\rangle} = l_{\langle 2\rangle} \wedge k_{\langle 1\rangle} = k_{\langle 2\rangle}\}$$

$$\text{if } h \geq k \text{ then } h := h - 1; l := l + k; \text{ then skip}$$

$$\{l_{\langle 1\rangle} = l_{\langle 2\rangle} \wedge k_{\langle 1\rangle} = k_{\langle 2\rangle}\}$$

Self-composition.

Darvas et al. and Barthe et al. realized that non-interference can be reduced to a safety property:

- Darvas, Hähnle and Sands: *A theorem proving approach to analysis of secure information flow.* WITS'03
- Barthe, D'Argenio and Rezk: *Secure Information Flow by Self-composition.* CSFW'04

$$\models \{\varphi\}\, P; P' \,\{\psi\} \qquad \Longrightarrow \qquad \models \{\varphi\} P \sim P' \{\psi\}$$

Ok for straight line code, but cumbersome otherwise.

Nick Benton's Relational Hoare Logic (RHL)

RHL

logical characterization and verification of static analyses and program optimizations

- verification of non-interference
- program analysis and verification
- non-relational Hoare Logic verification

The problem, for Benton:

> *commonly, analysis are intensionally expressed, whereas program transformations are enabled by more extensional interpretations.*

Benton's RHL

"At first sight, this may seem frighteningly simple-minded, but it actually works rather nicely."—$N.Benton$

$$\frac{\vdash C \sim C' : \Phi \wedge (B\langle 1\rangle \wedge B'\langle 2\rangle) \Rightarrow \Phi \wedge (B\langle 1\rangle = B'\langle 2\rangle)}{\vdash \texttt{while}\ B\ \texttt{do}\ C \sim \texttt{while}\ B'\ \texttt{do}\ C' : \Phi \wedge (B\langle 1\rangle = B'\langle 2\rangle) \Rightarrow \Phi \wedge \texttt{not}(B\langle 1\rangle \vee B'\langle 2\rangle)}$$

$$\frac{\vdash C \sim C' : \Phi \wedge (B\langle 1\rangle \wedge B'\langle 2\rangle) \Rightarrow \Phi' \quad \vdash D \sim D' : \Phi \wedge \texttt{not}(B\langle 1\rangle \vee B'\langle 2\rangle) \Rightarrow \Phi'}{\vdash \texttt{if}\ B\ \texttt{then}\ C\ \texttt{else}\ D \sim \texttt{if}\ B'\ \texttt{then}\ C'\ \texttt{else}\ D' : \Phi \wedge (B\langle 1\rangle = B'\langle 2\rangle) \Rightarrow \Phi'}$$

$$\vdash X \texttt{ := } E \sim Y \texttt{ := } E' : \Phi[E\langle 1\rangle/X\langle 1\rangle, E'\langle 2\rangle/Y\langle 2\rangle] \Rightarrow \Phi$$

$$\frac{\vdash C \sim C' : \Phi \Rightarrow \Phi' \quad \vdash D \sim D' : \Phi' \Rightarrow \Phi''}{\vdash C\ \texttt{;}\ D \sim C'\ \texttt{;}\ D' : \Phi \Rightarrow \Phi''}$$

Benton's RHL

Succeeds to verify some code optimizations and secure information flow.

Verified slicing example

$$\{\mathsf{true}\}$$

```
I := 1;                  I := 1;
S := 0;
P := 1;                  P := 1
while I<N do (    ==>    while I<N do (
 S := S+I;
 P := P*I;                P := P*I;
 I := I+1;)               I := I+1;)
```

$$\{I_{\langle 1\rangle} = I_{\langle 2\rangle} \wedge P_{\langle 1\rangle} = P_{\langle 2\rangle}\}$$

with loop inv. $\{I_{\langle 1\rangle} = I_{\langle 2\rangle} \wedge P_{\langle 1\rangle} = P_{\langle 2\rangle}\}$

Benton's RHL

But RHL fails to verify a simple example when the number of iterations does not coincide:

Consider e.g. the following basic example:

```
i := 0;                    j := 1;
while (i ≤ N) do           while (j ≤ N) do
    x += i;                    y += j;
    i++                        j++
```

Moreover, as opposed to traditional verification, this relational method has no tool support

Other syntactic methods for deriving program equivalences

- Hongseok Yang's relational separation logic
  - it confines reasoning to structurally equivalent programs
  - not implemented (except Crespo's Coq formalization).

- Santiago Zanella et. al. probabilistic relational Hoare logic.
  - focused on (but not limited to) probabilistic programs
  - extension in progress, but far from a realistic programming language

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Combine the best of self-composition and RHL, and enable the use of existing tools.

Program products

Main Idea

Given two programs c_1 and c_2 and post and pre relations φ and ψ define a program $c_1 \times c_2$ and pre and postconditions $\bar{\varphi}$ and $\bar{\psi}$ such that:

$$\text{if} \models \{\bar{\varphi}\}\, c_1 \times c_2\, \{\bar{\psi}\} \text{ then} \models \{\varphi\} c_1 \sim c_2 \{\psi\}$$

Not really a new idea!

Self-composition does essentially this, by setting $c_1 \times c_2 = c_1; c_2$.

- \+ Does not require programs to be structurally equivalent.
- \+ (Relatively) Complete.
- − Impractical: not really amenable to program verification (lacks the synchronized loop invariants of RHL).

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+ Does not require programs to be structurally equivalent.
+ (Relatively) Complete.
– Impractical: not really amenable to program verification (lacks the synchronized loop invariants of RHL).

Consider again the following basic example:

```
$i := 0$;
while ($i \leq N$) do
    $x$ += $i$;
    $i$++
```

```
$j := 1$;
while ($j \leq N$) do
    $y$ += $j$;
    $j$++
```

Self-composition as product construction:

```
$i := 0$;
while ($i \leq N$) do
    $x$ += $i$;
    $i$++
```

×

```
$j := 1$;
while ($j \leq N$) do
    $y$ += $j$;
    $j$++
```

⟶

```
$i := 0$;
while ($i \leq N$) do
    $x$ += $i$;
    $i$++;
$j := 1$;
while ($j \leq N$) do
    $y$ += $j$;
    $j$++
```

The verification of the code on the right requires providing a couple of quadratic loop invariants.

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$i := 0$;
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    $i$++;
$j := 1$;
while ($j \le N$) do
    $y$ += $j$;
    $j$++
```

The verification of the code on the right requires providing a couple of quadratic loop invariants.

A more clever product construction:

```
$i := 0$;
while ($i \le N$) do
    $x$ += $i$;
    $i$++
```

×

```
$j := 1$;
while ($j \le N$) do
    $y$ += $j$;
    $j$++
```

⟶

```
$i := 0$;
assert($i \le N$);
$x$ += $i$;  $i$++;
$j := 1$;
assert($i \le N \Leftrightarrow j \le N$);
while ($i \le N$) do
    $y$ += $j$;  $j$++;
    $x$ += $i$;  $i$++;
    assert($i \le N \Leftrightarrow j \le N$);
```

- The verification of the code on the right only needs the invariant $x = y \wedge i = j$.
- Relational verification is reduced to derivation of a product program followed by standard verification.

A more clever product construction:

```
$i := 0$;
while ($i \leq N$) do
    $x$ += $i$;
    $i$++
```

×

```
$j := 1$;
while ($j \leq N$) do
    $y$ += $j$;
    $j$++
```

$\longrightarrow$

```
$i := 0$;
assert($i \leq N$);
$x$ += $i$;  $i$++;
$j := 1$;
assert($i \leq N \Leftrightarrow j \leq N$);
while ($i \leq N$) do
    $y$ += $j$;  $j$++;
    $x$ += $i$;  $i$++;
    assert($i \leq N \Leftrightarrow j \leq N$);
```

- The verification of the code on the right only needs the invariant $x = y \wedge i = j$.
- Relational verification is reduced to derivation of a product program followed by standard verification.

Contribution:

- Syntactic method for constructing products that soundly abstract the behaviour of its constituents.
- Relational loop invariants become greatly simpler.
- A wide variety of application, e.g., correctness of advanced loop optimisations. Most of them verified with the Why verification framework, using SMT solvers and the Coq proof assistant.

Product construction rules

$$\frac{}{c_1 \times c_2 \to c_1; c_2} \qquad \frac{c_1 \times c_2 \to c \qquad c_1' \times c_2' \to c'}{c_1; c_1' \times c_2; c_2' \to c; c'}$$

$$\frac{c_1 \times c_2 \to c}{\mathsf{while}\ b_1\ \mathsf{do}\ c_1 \times \mathsf{while}\ b_2\ \mathsf{do}\ c_2 \to \mathsf{assert}(b_1 \Leftrightarrow b_2); \mathsf{while}\ b_1\ \mathsf{do}\ (c; \mathsf{assert}(b_1 \Leftrightarrow b_2))}$$

$$\frac{c_1 \times c_2 \to c \qquad c_1' \times c_2' \to c'}{\mathsf{if}\ b_1\ \mathsf{then}\ c_1\ \mathsf{else}\ c_1' \times \mathsf{if}\ b_2\ \mathsf{then}\ c_2\ \mathsf{else}\ c_2' \to \mathsf{assert}(b_1 = b_2); \mathsf{if}\ b_1\ \mathsf{then}\ c\ \mathsf{else}\ c'}$$

$$\frac{c_1 \succcurlyeq c_1' \qquad c_2 \succcurlyeq c_2' \qquad c_1' \times c_2' \to c}{c_1 \times c_2 \to c}$$

$$\ldots$$

Product construction rules

$$\frac{}{c_1 \times c_2 \rightarrow c_1; c_2} \qquad \frac{c_1 \times c_2 \rightarrow c \qquad c_1' \times c_2' \rightarrow c'}{c_1; c_1' \times c_2; c_2' \rightarrow c; c'}$$

$$\frac{c_1 \times c_2 \rightarrow c}{\text{while } b_1 \text{ do } c_1 \times \text{while } b_2 \text{ do } c_2 \rightarrow \text{assert}(b_1 \Leftrightarrow b_2); \text{while } b_1 \text{ do } (c; \text{assert}(b_1 \Leftrightarrow b_2))}$$

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$$\frac{c_1 \succcurlyeq c_1' \qquad c_2 \succcurlyeq c_2' \qquad c_1' \times c_2' \rightarrow c}{c_1 \times c_2 \rightarrow c}$$

. . .

Syntactic Reduction Rules

$$\frac{}{\text{if } b \text{ then } c_1 \text{ else } c_2 \succcurlyeq \text{assert}(b \Leftrightarrow \neg b'); \text{if } b' \text{ then } c_2 \text{ else } c_1}$$

$$\frac{}{\text{while } b \text{ do } c \succcurlyeq \text{assert}(b); c; \text{while } b \text{ do } c}$$

$$\frac{c_1 \succcurlyeq c_1' \qquad c_2 \succcurlyeq c_2'}{\text{if } b \text{ then } c_1 \text{ else } c_2 \succcurlyeq \text{if } b \text{ then } c_1' \text{ else } c_2'}$$

$$\frac{c \succcurlyeq c'}{\text{while } b \text{ do } c \succcurlyeq \text{while } b \text{ do } c'} \qquad \frac{c \succcurlyeq c' \qquad c' \succcurlyeq c''}{c \succcurlyeq c''} \qquad \frac{}{c \succcurlyeq c} \qquad \frac{c_1 \succcurlyeq c_1' \qquad c_2 \succcurlyeq c_2'}{c_1; c_2 \succcurlyeq c_1'; c_2'}$$

. . .

Main Results

- (Embedding of relational Hoare logic into standard one). For all derivations in relational hoare logic $\vdash \{\varphi\} c_1 \sim c_2 \{\psi\}$ there exist c such that $c_1 \times c_2 \rightarrow c$ and $\vdash \{\varphi\}\, c\, \{\psi\}$.
- (Soundness of the method). For all statements c_1 and c_2 and pre and post-relations φ and ψ, if $c_1 \times c_2 \rightarrow c$, $\models \{\varphi\}\, c\, \{\psi\}$ and φ is strong enough to ensure that c does not get stuck in any assert then $\models \{\varphi\} c_1 \sim c_2 \{\psi\}$.

THERE'S NO MAGIC HERE. PRODUCT CONSTRUCTION STILL DEMANDS A LOT OF EFFORT.

bubblesort continuity

Source code:

```
$i := 0$;
while ($i < N$) do
    $j := N-1$;
    while ($j > i$) do
        if ($a[j-1] > a[j]$) then
            $swap(a, j, j-1)$;
        $j--$
    $i++$
```

Program product (simplified):

```
$i := 0$; $i' := 0$;
while ($i < N$) do
    $j := N-1$; $j' := N-1$;
    while ($j > i$) do
        if ($a[j-1] > a[j]$) then
            $swap(a, j, j-1)$;
        if ($a'[j'-1] > a'[j']$) then
            $swap(a', j', j'-1)$;
        $j--$; $j'--$
    $i++$; $i'++$
```

$$\vDash \{\forall i.\ |a[i]-a'[i]| < \epsilon\}\ \mathbf{P}\ \{\forall i.\ |a[i]-a'[i]| < \epsilon\}$$

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        if ($a'[j'-1] > a'[j']$) then
            $swap(a', j', j'-1)$;
        $j--$;  $j'--$
    $i++$;  $i'++$
```

$$\models \{\forall i.\ |a[i]-a'[i]| < \epsilon\}\ \mathbf{P}\ \{\forall i.\ \ |a[i]-a'[i]| < \epsilon\}$$

Non-interference

```
$i := 0$;
while $(i < N)$ do
    if $(ps[i].JoinInd)$ then
        $j := 0$;
        while $(j < M)$ do
            if $(ps[i].PID = es[j].EID)$ then
                $tab[i].employee := es[j]$;
                $tab[i].payroll := ps[i]$;
            $j$++;
    $i$++;
```

Pre

$es = es' \land \forall i : 0 \leq i < N : ps[i].PID = ps'[i].PID \land$
$ps[i].JoinInd = ps'[i].JoinInd \land (ps[i].JoinInd \Rightarrow ps[i].salary = ps'[i].salary)$

Post

$\forall i : 0 \leq i < N : ps[i].JoinInd \Rightarrow tab[i] = tab'[i]$

Non-interference

Program product verification:

```
$\{Pre : es = es' \wedge \forall i : 0 \leq i < N : ps[i].PID = ps'[i].PID \wedge$
    $ps[i].JoinInd = ps'[i].JoinInd \wedge (ps[i].JoinInd \Rightarrow ps[i].salary = ps'[i].salary)\}$
  $i := 0$; $i' := 0$; assert($i < N \Leftrightarrow i' < N$);
  while ($i < N$) do
    assert($ps[i].JoinInd \Leftrightarrow ps'[i'].JoinInd$);
    if ($ps[i].JoinInd$) then
      $j := 0$; $j' := 0$; assert($j < M \Leftrightarrow j' < M$);
      while ($j < M$) do
        assert($ps[i].PID = es[j].EID \Leftrightarrow ps'[i'].PID = es'[j'].EID$);
        if ($ps[i].PID = es[j].EID$) then
          $tab[i].employee := es[j]$; $tab'[i'].employee := es'[j]$;
          $tab[i].payroll := ps[i]$; $tab'[i'].payroll := ps'[i]$;
        $j{+}{+}$; $j'{+}{+}$; assert($i < N \Leftrightarrow i' < N$);
    $i{+}{+}$; $i'{+}{+}$; assert($i < N \Leftrightarrow i' < N$);
$\{Post : \forall i : 0 \leq i < N : ps[i].JoinInd \Rightarrow tab[i] = tab'[i]\}$
```

loop pipelining

Source program:

```
$i := 0$;
while ($i < N$) do
    $a[i]$++;  $b[i]$ += $a[i]$;
    $c[i]$ += $b[i]$;  $i$++
```

Transformed program:

```
$j := 0$;
$\bar{a}[0]$++;  $\bar{b}[0]$ += $\bar{a}[0]$;
$\bar{a}[1]$++;
while ($j < N-2$) do
    $\bar{a}[j+2]$++;
    $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
    $\bar{c}[j]$ += $\bar{b}[j]$;  $j$++
$\bar{c}[j]$ += $\bar{b}[j]$;
$\bar{b}[j+1]$ += $\bar{a}[j+1]$;
$\bar{c}[j+1]$ += $\bar{b}[j+1]$
```

Product program:

```
$\{a = \bar{a} \wedge b = \bar{b} \wedge c = \bar{c}\}$
  $i := 0$;
  $j := 0$;
  assert($i < N$);
  $a[i]$++;  $b[i]$ += $a[i]$;
  $c[i]$ += $b[i]$;  $i$++
  $\bar{a}[0]$++;  $\bar{b}[0]$ += $\bar{a}[0]$;
  assert($i < N$);
  $a[i]$++;  $b[i]$ += $a[i]$;
  $c[i]$ += $b[i]$;  $i$++
  $\bar{a}[1]$++;
  assert($i < N \Leftrightarrow j < N-2$);
  while ($i < N$) do
      $a[i]$++;  $b[i]$ += $a[i]$;
      $c[i]$ += $b[i]$;  $i$++
      $\bar{a}[j+2]$++;
      $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
      $\bar{c}[j]$ += $\bar{b}[j]$;  $j$++
      assert($i < N \Leftrightarrow j < N-2$);
  $\bar{c}[j]$ += $\bar{b}[j]$;
  $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
  $\bar{c}[j+1]$ += $\bar{b}[j+1]$
$\{a = \bar{a} \wedge b = \bar{b} \wedge c = \bar{c}\}$
```

loop pipelining

Source program:

```
$i := 0$;
while $(i < N)$ do
    $a[i]$++;  $b[i]$ += $a[i]$;
    $c[i]$ += $b[i]$;  $i$++
```

Transformed program:

```
$j := 0$;
$\bar{a}[0]$++;  $\bar{b}[0]$ += $\bar{a}[0]$;
$\bar{a}[1]$++;
while $(j < N-2)$ do
    $\bar{a}[j+2]$++;
    $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
    $\bar{c}[j]$ += $\bar{b}[j]$;  $j$++
$\bar{c}[j]$ += $\bar{b}[j]$;
$\bar{b}[j+1]$ += $\bar{a}[j+1]$;
$\bar{c}[j+1]$ += $\bar{b}[j+1]$
```

Product program:

```
$\{a = \bar{a} \wedge b = \bar{b} \wedge c = \bar{c}\}$
  $i := 0$;
  $j := 0$;
  assert$(i < N)$;
  $a[i]$++;  $b[i]$ += $a[i]$;
  $c[i]$ += $b[i]$;  $i$++
  $\bar{a}[0]$++;  $\bar{b}[0]$ += $\bar{a}[0]$;
  assert$(i < N)$;
  $a[i]$++;  $b[i]$ += $a[i]$;
  $c[i]$ += $b[i]$;  $i$++
  $\bar{a}[1]$++;
  assert$(i < N \Leftrightarrow j < N-2)$;
  while $(i < N)$ do
      $a[i]$++;  $b[i]$ += $a[i]$;
      $c[i]$ += $b[i]$;  $i$++
      $\bar{a}[j+2]$++;
      $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
      $\bar{c}[j]$ += $\bar{b}[j]$;  $j$++
      assert$(i < N \Leftrightarrow j < N-2)$;
  $\bar{c}[j]$ += $\bar{b}[j]$;
  $\bar{b}[j+1]$ += $\bar{a}[j+1]$;
  $\bar{c}[j+1]$ += $\bar{b}[j+1]$
$\{a = \bar{a} \wedge b = \bar{b} \wedge c = \bar{c}\}$
```

static-caching (Annie Liu)

Source program:

```
$i_1 := 0$;
while ($i_1 \leq N - M$) do
  $s[i_1] := 0$;  $k_1 := 0$;
  while ($k_1 \leq M - 1$) do
    $l_1 := 0$;
    while ($l_1 \leq L - 1$) do
      $s[i_1]$+=$a[i_1 + k_1, l_1]$;  $l_1$++;
    $k_1$++;
  $i_1$++
```

Transformed program:

```
$t[0] := 0$;  $k_2 := 0$;
while ($k_2 \leq M - 1$) do
  $b[k_2] := 0$;  $l_2 := 0$;
  while ($l_2 \leq L - 1$) do
    $b[k_2]$ += $a[k_2, l_2]$;  $l_2$++;
  $t[0]$ += $b[k_2]$;  $k_2$++;
$i_2 := 1$;
while ($i_2 \leq N - M$) do
  $b[i_2 + M - 1] := 0$;  $l_2 := 0$;
  while ($l_2 \leq L - 1$) do
    $b[i_2 + M - 1]$+=$a[i_2 + M - 1, l_2]$;  $l_2$++;
  $z := b[i_2 + M - 1] - b[i_2 - 1]$;
  $t[i_2] := t[i_2 - 1] + z$;  $i_2$++
```

static-caching (Annie Liu)

```
{true}
$i_1 := 0$; assert($i_1 \le N-M$); $s[i_1] := 0$; $k_1 := 0$; $t[0] := 0$; $k_2 := 0$;
assert($k_1 \le M-1 \Leftrightarrow k_2 \le M-1$);
while ($k_1 \le M-1$) {$Inv_1$} do
    $l_1 := 0$; $b[k_2] := 0$; $l_2 := 0$; assert($l_1 \le L-1 \Leftrightarrow l_2 \le L-1$);
    while ($l_1 \le L-1$) {$Inv_2$} do
        $s[i_1]$+=$a[i_1+k_1, l_1]$; $l_1$++; $b[k_2]$ += $a[k_2, l_2]$; $l_2$++;
        assert($l_1 \le L-1 \Leftrightarrow l_2 \le L-1$);
    $k_1$++; $t[0]$ += $b[k_2]$; $k_2$++; assert($k_1 \le M-1 \Leftrightarrow k_2 \le M-1$);
$i_1$++; $i_2 := 1$; assert($i_1 \le N-M \Leftrightarrow i_2 \le N-M$);
while ($i_1 \le N-M$) {$Inv_3$} do
    $b[i_2+M-1] := 0$; $l_2 := 0$;
    while ($l_2 \le L-1$) {$Inv_4$} do
        $b[i_2+M-1]$+=$a[i_2+M-1, l_2]$; $l_2$++;
    $z := b[i_2+M-1] - b[i_2-1]$; $t[i_2] := t[i_2-1]+z$; $i_2$++;
    $s[i_1] := 0$; $k_1 := 0$;
    while ($k_1 \le M-1$) {$Inv_5$} do
        $l_1 := 0$;
        while ($l_1 \le L-1$) {$Inv_6$} do
            $s[i_1]$+=$a[i_1+k_1, l_1]$; $l_1$++;
        $k_1$++;
    $i_1$++
    assert($i_1 \le N-M \Leftrightarrow i_2 \le N-M$);
{$\forall\, i \in [0, N-M].\ s[i] = t[i]$}
```

$$Inv_2 \doteq i_1{=}0 \wedge k_1{=}k_2 \wedge l_1{=}l_2 \wedge k_1{\le}M \wedge l_1{\le}L \wedge s[i_1] = t[0]+b[k_1] = \textstyle\sum_{k'=0}^{k_1-1} b[k'] + b[k_1] \wedge$$

$$\forall\, k' \in [0, k_1).\ b[k'] = \textstyle\sum_{l'=0}^{L-1} a[k', l'] \wedge b[k_1] = \textstyle\sum_{l'=0}^{l_1-1} a[k_1, l']$$

$$Inv_3 \doteq i_1 = i_2 \wedge i_1 \le N-M+1 \wedge \forall\, i' \in [0, i_1) \Rightarrow s[i'] = t[i'] = \textstyle\sum_{k'=0}^{M-1} b[k'+i'] \wedge \forall\, i' \in [0, i_1+M-1).\ b[i'] = \textstyle\sum_{l'=0}^{L-1} a[i', l']$$

$$Inv_4 \doteq Inv_3 \wedge k_1 \le M \wedge l_2 \le L \wedge b[i_2+M-1] = \textstyle\sum_{l'=0}^{l_2-1} a[i_2+M-1, l'] \wedge s[i_1] = \textstyle\sum_{k'=0}^{k_1-1} b[k'+i_1]$$

$$Inv_6 \doteq Inv_3 \wedge k_1 \le M \wedge l_1 \le L \wedge b[i_2+M-1] = \textstyle\sum_{l'=0}^{L-1} a[i_2+M-1, l'] \wedge s[i_1] = \textstyle\sum_{k'=0}^{k_1-1} b[k'+i_1] + \textstyle\sum_{l'=0}^{l_1-1} a[i_1+k_1, l']$$

Overview of Why Verification

Optimisation	Proof Obligations	Automatically Discharged
Strength Reduction	9	9
Loop Reversal	13	13
Loop Interchange	41	38
Loop Alignment	53	53
Loop Pipelining	77	77
Static Caching	188	165

Open Research Topics

- Advanced heuristics for the construction of program products.
- Tool for interactive (i.e. tactic based) construction of products.
- Extension of the method for *k*-safety properties.
- Separation on heap manipulating programs (there are solutions to this, as relocatable programs, but never implemented).
- Program separation and Hongseok Yang's Relational Separation logics. (there are discussions with Bart Jacobs towards the *Multi*VeriFast tool)
- Relational program synthesis.

Conclusion

- A technique to reduce relational program verification (2-safety properties) into standard ones.
- This technique is usually constrained to structurally similar programs or requires full-blown functional verification.
- We generalized a marriage of RHL with self-composition coping with non structurally equivalent programs.
- Enables using existing program verification tools for such verification problems
- Requires ingenious effort to construct the program product
- We have illustrated the usefulness of our approach by using it to validate advanced loop optimization.