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CONNECTING BILATTICES THEORY AND FUZZY LOGIC

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AIM: extend the tools, the framework and the apparatus of fuzzy logic

Meta-logic programming for a synonymy logic

Connections between bilattices theory and fuzzy logic

Fuzzy model theory

Bilattices theory and fuzzy logic

AIM: investigate the potentialities of bilattice theory for the graded approach to fuzzy logic

Bilattices as tool

Manage positive and negative information

Treat incomplete and inconsistent information

IDEA: interpret the elements of a bilattice *B* as pieces of information on the elements in *V*.

A *valuation structure is* a complete lattice $V = (V, \leq, 0, 1)$ with $0 \neq 1$.

A *bilattice* (Ginsberg) is a structure with two bounded lattice orders

$$B=(B, \leq_t, \leq_k, \text{False}, \text{True}, \bot, \top)$$

The order $\leq_t$ is interpreted as degree of truth, the order $\leq_k$ is related with the amount of information or knowledge.

$$x \leq_t y \; ; \quad x \leq_k y$$

It is possible to define a negation with some related axioms

$$\mathcal{B}=(B, \leq_t, \leq_k, \sim, False, True, \bot, \top)$$

1. $x \leq_t y \Rightarrow \sim y \leq_t \sim x$
2. $x \leq_k y \Rightarrow \sim x \leq_k \sim y$
3. $\sim\sim x = x$

The negation ~ is order-reversing with respect to $\leq_t$ and order-preserving with respect to $\leq_k$

Examples

Product bilattice

$$B(L) = (L \times L, \leq_t, \leq_k, \sim, (0,1), (1,0), (0,0), (1,1))$$

Where $(x, x') \leq_t (y, y') \Leftrightarrow x \leq y$ and $y' \leq x'$

$(x, x') \leq_k (y, y') \Leftrightarrow x \leq y$ and $x' \leq y'$

$\sim(x, x') = (x', x)$.

Interval bilattice

$$I(L)=(I(L), \leq_t, \leq_k, \{0\}, \{1\}, [0,1], \varnothing)$$

Where $\leq_k$ is the dual of inclusion

$\forall\ [a,b], [c,d] \in I(L)-\{\varnothing\}, [a,b] \leq_t [c,d] \Leftrightarrow a \leq c$ and $b \leq d$,

Semantics

To connect a bilattice *B* with a valuation structure *V* we need to define a relation $\models^*$ from *V* to *B*.
The meaning of $\lambda \models^* b$ is that *b* is a correct piece of information on λ

We call *bt-system* a structure $(V, B, \models^*)$ where *V* is a valuation structure, *B* is a complete bilattice and $\models^*$ is a relation in $V \times B$ such that:

- **i)** $\lambda \models^* x$ **and** $x' \leq_k x \Rightarrow \lambda \models^* x'$
- **ii)** $\forall \lambda \in V$ **the set** $\{x \in B : \lambda \models^* x\}$ **admits a k-sup**
- **iii)** $0 \models^*$ **False ; 1** $\models^*$ **True.**

For example, if *B* is a interval bilattice a possible definition is

$$\lambda \models^* [a_1,a_2] \Leftrightarrow \lambda \in [a_1,a_2]$$

Semantics

Extend some semantics notions of fuzzy logic to bilattice

We call *B*-subset of formulas or valuation any map $v : F \rightarrow B$.

A multi-valued semantics is a class *M* of maps $m : F \rightarrow V$ that are truth functional; the elements in *M* are called *models*.

We say that $m \in M$ is a *model of v*, in brief $m \models v$, if $m(\alpha) \models^{*} v(\alpha)$ for every formula α.

We define two different semantics, one of these is able to manage incomplete and inconsistent information.

Syntax

We define a notion of deduction apparatus for bilattice by extending the notion of fuzzy inference rule in a suitable way (in particular in the case of Kripke worlds-based bilattice)

Positive Modus Ponens

$$\left\langle \frac{\alpha \quad \alpha \rightarrow_t \beta}{\beta} \,\middle|\, \frac{(A_+, A_-) \quad (I_+, I_-)}{(A_+, A_-) \Diamond^+ (I_+, I_-)} \right\rangle$$

Negative Modus Ponens

$$\left\langle \frac{\alpha \quad \alpha \rightarrow_f \beta}{\beta} \,\middle|\, \frac{(A_+, A_-) \quad (I_+, I_-)}{(A_+, A_-) \Diamond^- (I_+, I_-)} \right\rangle$$

where

$\alpha \rightarrow_t \beta$ **is the formula** $\neg \alpha \vee \beta$;

$\alpha \rightarrow_f \beta$ **is the formula** $\neg \alpha \wedge \beta$;

$(A_+, A_-) \Diamond^+ (I_+, I_-) = (A_+ \cap I_+, \varnothing)$;

$(A_+, A_-) \Diamond^- (I_+, I_-) = (\varnothing, A_- \cap I_-)$

Equivalence between *K*-closed valuation an *W*-closed valuation

Theorem: v is a K-closed valuation $\Leftrightarrow$ v is a W-closed valuation

Completeness theorem for both semantics

Estension of Pavelka's approach to bilattice logic

D. Genito, G. Gerla, "An attempt to connect bilattice theory with fuzzy logic", submitted to *Archive for mathematical logic*

FUTURE WORKS

- **Extend our logical system to any bilattice**
- **Extend fuzzy control by bilattice theory**
- **Bilattice logic programming**
- **Bilattice logic with similarity**
- **Bilattice logic with T-norm**
- **Bilattice and Qualified Logic programming**