Equality Constraints on the Algebras of Finite and Infinite Trees

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Description

- Trees

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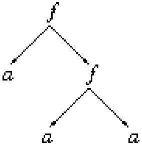

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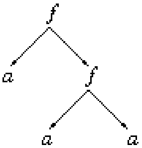

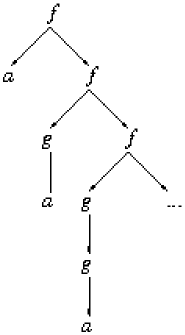

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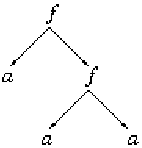

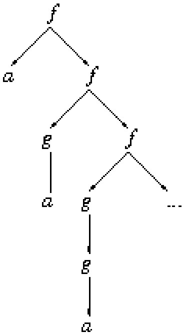

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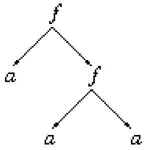

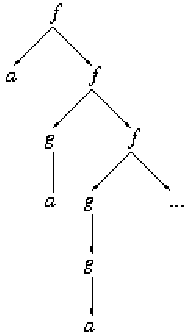

- Signature
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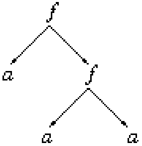
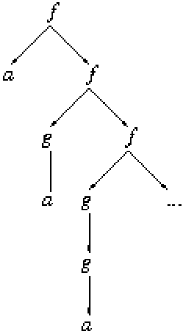
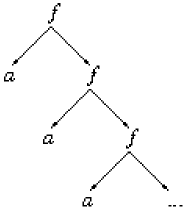




- Signature
- Algebra
- First-order constraints

Description II

- Rational trees



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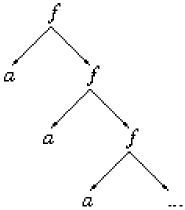

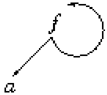

Main Objectives

- Provide an axiomatization

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- Prove its completeness

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- Prove its completeness
- Efficiency

Motivation

- Logic programming and negation: unification vs. disunification

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- Knowledge representation
- Reasoning

Results

- The theories of finite/infinite trees are complete (Maher)
  - Decision procedure for finite trees (Comon&Lescanne)
  - Decision procedure for finite/infinite trees and infinite signatures (Colmerauer *et al.*)

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- The theories of finite/infinite trees are complete (Maher)
  - Decision procedure for finite trees (Comon&Lescanne)
  - Decision procedure for finite/infinite trees and infinite signatures (Colmerauer *et al.*)
- The decision problem is non-elementary (Vorobyov)

Proposals

- Prefix normal form

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- Prefix normal form
  - Solved-form

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- Prefix normal form
    - Solved-form
- Quantifier elimination

This Talk

Infinite Trees

Axiomatization

1. For every $f \in \Sigma$

$$\forall \overline{x} \forall \overline{y} \ (\ f(\overline{x}) \approx f(\overline{y}) \ \leftrightarrow \ \overline{x} \approx \overline{y} \ )$$

2. For every $f, g \in \Sigma$, $f \neq g$

$$\forall \overline{x} \forall \overline{y} \ (\ f(\overline{x}) \not\approx g(\overline{y}) \ )$$

3. For every rational solved form $\overline{x} \approx \overline{t}[\overline{x} \cdot \overline{y}]$ (unique solution)

$$\forall \overline{y} \exists ! \overline{x} \ (\ \overline{x} \approx \overline{t}[\overline{x} \cdot \overline{y}] \ )$$

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$$\forall \overline{y} \forall \overline{z} \exists \overline{x} \; (\; \overline{x} \approx \overline{t}[\overline{x} \cdot \overline{y}] \wedge \; (\; \overline{z} \approx \overline{t}[\overline{z} \cdot \overline{y}] \rightarrow \overline{x} \approx \overline{z} \;) \;)$$

Independence Property of Disequations

For every system of equations $\phi = \overline{x} \approx \bar{t}$ and every conjunction of disequations $\psi = \bigwedge_{i=1}^{n} s_i \not\approx r_i$

$$\phi \wedge \psi \text{ is satisfiable} \iff \phi \wedge s_i \not\approx r_i \text{ is individually satisfiable for each } 1 \leq i \leq n$$

Normal Form

A basic formula for $\overline{x}$ is either $true$, $false$ or a formula $\exists\overline{w}\ c(\overline{x},\overline{w})$ such that

$$c(\overline{x},\overline{w}) = \overline{x}\cdot\overline{w} \approx \overline{t}[\overline{x}\cdot\overline{w}] \wedge \bigwedge_{i=1}^{m} \forall\overline{v}^i\ z_i \not\approx s_i[\overline{x}\cdot\overline{w}\cdot\overline{v}^i]$$

where $z_i \in \overline{x}\cdot\overline{w}$

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Examples:

$$x \not\approx y \quad \mapsto \quad \exists \overline{w}\ (\ x \approx w_1 \wedge y \approx w_2 \wedge w_1 \not\approx w_2\ )$$

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$$x \approx s(x) \wedge y \approx s(y) \wedge x \not\approx y$$

Elimination of Quantifiers

$$\forall x \; \exists y \; [\; \exists \overline{w} \; (\; x \approx w_1 \wedge y \approx w_2 \wedge w_1 \not\approx w_2 \;) \;]$$

Elimination of Quantifiers

$$\begin{aligned} &\forall x\, \exists y\, [\, \exists \overline{w}\, (\, x \approx w_1 \wedge y \approx w_2 \wedge w_1 \not\approx w_2\,)\,] \\ &\qquad \equiv \quad \forall x\, [\, \exists \overline{w}\, (\, x \approx w_1 \wedge w_1 \not\approx w_2\,)\,] \end{aligned}$$

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$$\equiv\quad \forall x\ [\ \exists \overline{w}\ (\ x \approx w_1 \wedge w_1 \not\approx w_2\ )\ ]$$
$$\equiv\quad \forall x\ [\ \exists w_1\ (\ x \approx w_1\ )\ ]$$

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\begin{aligned}
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\equiv \quad & \forall x\ [\ \exists w_1\ (\ x \approx w_1\ )\ ] \\
\equiv \quad & \neg \exists x\ [\ \neg \exists w_1\ (\ x \approx w_1\ )\ ]
\end{aligned}
$$

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&\quad \equiv \quad \forall x\ [\ \exists w_1\ (\ x \approx w_1\ )\ ] \\
&\quad \equiv \quad \neg \exists x\ [\ \neg \exists w_1\ (\ x \approx w_1\ )\ ] \\
&\quad \equiv \quad \neg \exists x\ [\ \forall w_1\ (\ x \not\approx w_1\ )\ ] \\
&\quad \equiv \quad \neg false \\
&\quad \equiv \quad true
\end{aligned}
$$

Finite Signatures

$$\exists w \; (\; x \approx w \wedge w \not\approx a \wedge \forall v \; w \not\approx g(v) \wedge \forall \overline{v} \; w \not\approx f(v_1, v_2) \;)$$

Axiomatization

4 Domain Closure Axiom or DCA:

$$\forall x \bigvee_{f \in \Sigma} \exists \overline{w}\ x \approx f(\overline{w})$$

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$$\forall \overline{x} \forall \overline{y} \ (\ f(\overline{x}) \not\approx g(\overline{y})\ )$$

3. For every term $t[x]$ containing x except x

$$\forall x \ (\ x \not\approx t[x]\ )$$

Solved Form

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Complexity and Efficiency Issues

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- In the case of finite signatures, we have to use DCA

Efficiency Issues

- Specific decision methods for particular classes of first-order constraints

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Efficiency Issues

- Specific decision methods for particular classes of first-order constraints
- More expressive notions of normal form
  - Represent more than one solution in a single constraint
  - Obtain a compact representation of formulas
- Other classical techniques:
  - Divide-and-conquer

Warning!!!

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- Handling formulas using expressive normal forms has to be efficient
    - Boolean operations: conjunction and negation
    - Satisfiability test
    - Elimination of existential variables

A First Approach

A) $$\overline{x} \approx \overline{t} \wedge \bigwedge_{w \in \overline{w}} \bigwedge_{i=1}^{m} \forall \overline{v}^i \; w \not\approx s_i[\overline{w} \cdot \overline{v}^i]$$

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B) $$\overline{x} \approx \overline{t} \wedge \bigwedge_{w \in \overline{w}} (\bigvee_{j=1}^{n} \exists \overline{z}^j \; w \approx r_j[\overline{z}^j] \bigwedge_{i=1}^{m} \forall \overline{v}^i \; w \not\approx s_i[\overline{w} \cdot \overline{v}^i] \;)$$

where each $r_j[\overline{z}^j]$ is linear

Compact Representation of Solutions

$$\exists w \; (\; x \approx f(w,a) \wedge \forall v \; (w \not\approx g(g(v))) \wedge \forall \overline{v} \; (w \not\approx f(f(v_1,v_2),g(v_3))) \;)$$

$$\Downarrow$$

$$
\begin{aligned}
& (\; x \approx f(a,a) \;) \;\vee\; (\; x \approx f(g(a),a) \;) \;\vee \\
& \exists \overline{y} \; (\; x \approx f(g(f(y_1,y_2)),a) \;) \;\vee \\
& (\; x \approx f(f(a,a),a) \;) \;\vee \\
& \exists \overline{y} \; (\; x \approx f(f(a,f(y_1,y_2)),a) \;) \;\vee \\
& \exists y \; (\; x \approx f(f(g(y),a),a) \;) \;\vee \\
& \exists \overline{y} \; (\; x \approx f(f(g(y_1),f(y_2,y_3)),a) \;)
\end{aligned}
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Satisfiability Test

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$$f(x_1, x_2)/\{f(w, w)\}$$

Satisfiability Test II

$$\underbrace{\overline{x} \approx \overline{t}}_{\text{Satisfiable}} \quad \wedge \quad \underbrace{\bigwedge_{w \in \overline{w}} \bigwedge_{i=1}^{m} \forall \overline{v}^i \; w \not\approx s_i[\overline{w} \cdot \overline{v}^i]}_{\text{May be unsatisfiable}}$$

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$$\overline{x} \approx \overline{t} \wedge \bigwedge_{w \in \overline{w}} \left(\bigvee_{j=1}^{n} \quad \underbrace{\exists \overline{z}^j \; w \approx r_j[\overline{z}^j]}_{\substack{\text{Initial} \\ \text{explicit representation}}} \quad \bigwedge_{i=1}^{m} \forall \overline{v}^i \; w \not\approx s_i[\overline{w} \cdot \overline{v}^i] \right)$$

A Second Approach

To use the notion of implicit representation as normal form

Conclusions

- Well-known problems

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- Study techniques to improve efficiency

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- Study techniques to improve efficiency
- Extension to other notions of equality